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Dynamic Analysis and Extinction Thresholds in a Stochastic SEIRS Model with Standard and Saturated Incidence Rates

Saravanan, S. [1] and Monica, C. * [1]

[1]*Department of Mathematics, SAS, Vellore Institute of Technology, Vellore-632 014, Tamil Nadu, India*

E-mail: monica.c@vit.ac.in
**Corresponding author*

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Abstract

This study examines a stochastic SEIRS epidemic model featuring bilinear standard and saturated incidence rates, providing a more precise depiction of the interactions between susceptible and infected individuals in the population. In order to establish the validity of our mathematical strategy, we first construct a pertinent Lyapunov function to demonstrate the existence and uniqueness of a positive global solution for this SEIRS model. Next, we explore the dynamic behavior of the stochastic SEIRS model, with a specific focus on the conditions that lead to certain long-term behaviors in the system. It is worth noting that the condition $\mathscr{R}_0^S > 1$ is enough to establish the presence of an ergodic stationary distribution, which signifies the stable persistence of the disease within the population. On the other hand, we show that the condition $\widehat{\mathscr{R}}_0^S < 1$ is essential for the disease to be eradicated, offering valuable insights into strategies for eliminating the disease. In order to validate our theoretical findings, we present a set of numerical simulations that serve to illustrate and verify our results. These simulations provide a concise and scholarly view of the theoretical constructs, emphasizing the practicality of our model in addressing real-world epidemics.

Keywords: extinction; Lyapunov function; stationary distribution; standard incidence rate; saturated incidence rate; stochastic SEIRS epidemic model.

1 Introduction

Recently, there has been a significant emphasis on studying the dynamic behavior of various epidemic models to control and manage the transmission of infectious diseases in fields such as population dynamics, biology, ecology, and environmental science. A growing body of literature suggests that concise models are a valuable tool for understanding the dynamics of infectious disease transmission. This concept was first introduced and studied by Kermack and McKendrick [10]. A generalized upper bound for the size of an epidemic can be derived for a graph in a specific homogeneous scenario at a fixed time $t > 0$. Gray et al. [6] presented a concise and scholarly analysis of a stochastic SIS model, investigating the impact of changes in the contact rate. Hussain et al. [8] investigated the stochastic permanence in an epidemic model with a standard incidence rate, focusing on a concise and academic analysis. A study by Cao et al. [1] explored the dynamic behavior of a two-group stochastic SIRS epidemic model with standard incidence rates. The stationary distribution and extinction of the stochastic SIRS epidemic model were critically examined by Lahrouz and Omari [13].

The study conducted by Yang et al. [25] provides evidence that stochastically perturbed SIR and SEIR epidemic models with saturated incidence exhibit ergodicity and extinction. In their study, Din et al. [2] investigated various aspects of a stochastic Hepatitis B epidemic model with partial immunity, including the stationary distribution, extinction, and optimal control. Their research provided concise and academic insights into this complex topic. Liu [21] studied the dynamics of positive solutions to SIR and SEIR epidemic models, focusing on saturated incidence rates. In a study conducted by Lei et al. [14], it was discovered that the stochastic SIRI epidemic model exhibits dynamic behaviors that result in a stationary distribution and eventual extinction. Liu et al. [19] analyzed stationary distributions and extinctions in a stochastic SIRI model with relapse. Zhang et al. [27] proposed a concise and academic stochastic SIQR model that incorporates both stationary distribution and extinction. A stationary distribution was found in a stochastic SEIQ model with transitory immunity by Zhang et al. [29]. A study conducted by Gokila et al. [5] examined the stationary distribution and global stability of a stochastic predator-prey model that incorporates disease in prey populations.

After a comprehensive literature review, it is evident that most researchers have focused on either standard or saturated incidence rates to understand the dynamics of immunity loss among recovered individuals. This paper, however, takes a novel approach by investigating the combined effects of both standard and saturated incidence rates. Our aim is to determine whether integrating these two incidence types can more effectively manage the decline in immunity and subsequently control disease spread. It is crucial to highlight that environmental white noise plays a significant role in this context. The stochastic perturbations introduced by white noise can influence disease dynamics, potentially steering the process toward desirable outcomes. By incorporating environmental noise effects, this study aims to provide a more robust and realistic model for understanding and controlling the spread of the disease. Ultimately, our research seeks to offer deeper insights into the mechanisms of disease transmission and control, potentially leading to more effective strategies for managing epidemics.

Nowadays, many researchers are focusing on the analysis and development of SEIR models to better understand epidemic dynamics [20, 15]. These studies explore various aspects such as transmission pathways, latency periods, and the influence of control strategies. Further advancements have been made by incorporating additional complexities into the SEIR framework, including stochastic effects, treatment interventions, and policy-driven measures [18, 4]. Such extensions contribute to a more comprehensive understanding of disease behavior and support the development of more effective mitigation strategies. The standard incidence of the deterministic

SEIR model can be described as follows [21]:

$$\begin{aligned}
\frac{d\mathcal{S}}{dt} &= \mathscr{A} - \frac{\beta \mathcal{S} I}{\mathscr{N}} - \mu \mathcal{S}, \\
\frac{d\mathcal{E}}{dt} &= \frac{\beta \mathcal{S} I}{\mathscr{N}} - (\phi + \mu)\mathcal{E}, \\
\frac{dI}{dt} &= \phi \mathcal{E} - (\pi + \mu + \alpha) I, \\
\frac{d\mathcal{R}}{dt} &= \pi I - \mu \mathcal{R}.
\end{aligned} \tag{1}$$

Understanding and identifying the specific factors involved in periodic outbreaks is crucial for predicting and preventing the spread of infectious diseases [6]. Traditional models often focus on the transition of individuals from susceptibility to exposure, then to the infected stage, and finally to recovery. However, these models frequently overlook the critical issue of immunity loss among recovered individuals. Although recovered individuals may revert to susceptibility and aid in the transmission of the infection, the possible ramifications of this loss of immunity are not sufficiently taken into account. It is essential to recognize that the immunity of recovered individuals can deteriorate, complicating the dynamics of disease transmission.

To address this gap, the proposed system emphasizes the importance of monitoring recovered individuals to prevent them from re-entering the susceptible stage. The SEIR model posits that individuals who have undergone treatment might lose their immunity over time, making them vulnerable to reinfection and capable of re-entering the susceptible population due to reduced immunity. By incorporating the concept of Brownian motion, where $v > 0$ indicates a loss of immunity at a fixed time $t > 0$, and considering a network with homogeneous special cases, the model aims to provide a more accurate representation of disease dynamics. This enhanced SEIR model can be utilized to comprehend the transmission dynamics of various infectious diseases, including measles, AIDS, dengue, and H7N9. Thus, in order to more accurately predict and manage the development of infectious diseases, this method emphasizes how important it is to incorporate immunity loss into epidemic models.

2 Model Description

At time t, the total population $\mathcal{N}(t)$ is segregated into four categories, which are people who are susceptible $\mathcal{S}(t)$, people who were exposed $\mathcal{E}(t)$, people with the infection $I(t)$, and recovered people $\mathcal{R}(t)$. The epidemic models are susceptible-exposed-infection-recovered-susceptible ($\mathcal{SEIRS}$) with $g(I)\mathcal{S}$ as the saturated incidence rate. When I increase in size, $g(I)$ is tends to a saturation level [17],

$$g(I)\mathcal{S} = \frac{\beta_1 \mathcal{S} I}{1 + \alpha I}.$$

Assume that $\beta_1 I$ and $\beta_2 \mathcal{E}$ represent the infection force and exposed force of the disease whereas $\dfrac{1}{1+\alpha I}$ indicates the inhibition effect brought about by the change in behavior of the susceptible individuals during the increase in numbers or the crowding effect brought about by the infective

individuals. A deterministic $\mathcal{SEIRS}$ model with standard incidence may be described as follows:

$$\begin{aligned}
\frac{d\mathcal{S}}{dt} &= \lambda - \frac{\beta_1 \mathcal{S} I}{1+\alpha I} - \beta_2 \mathcal{S}\mathcal{E} - \mu\mathcal{S} + \delta\mathcal{R},\\
\frac{d\mathcal{E}}{dt} &= \frac{\beta_1 \mathcal{S} I}{1+\alpha I} + \beta_2 \mathcal{S}\mathcal{E} - (\phi+\mu)\mathcal{E},\\
\frac{dI}{dt} &= \phi\mathcal{E} - (\pi+\mu+\theta) I,\\
\frac{d\mathcal{R}}{dt} &= \pi I - (\mu+\delta)\mathcal{R}.
\end{aligned} \tag{2}$$

A detailed list of the model parameters along with their respective descriptions and units is presented in Table 1.

Table 1: Model parameter and description.

Parameter	Description	Units
λ	Birth rate	Individual/day
β_1, β_2	Transmission co-efficients	$(\text{Individual}\times \text{ day})^{-1}$
μ	Rate of natural mortality	day^{-1}
ϕ	Rate at which exposed individuals develop infectiousness	day^{-1}
π	Recovery rate for infected individuals	day^{-1}
θ	Disease induced death rate	day^{-1}
δ	Recovered people lost their immunity rate	day^{-1}
α	Saturation incidence of inhibition effect rate	day^{-1}

According to the above model, the reproductive number is $\mathcal{R}_0 = \dfrac{\beta_1\phi\lambda}{\mu(\mu+\phi)(\pi+\mu+\theta)} + \dfrac{\beta_2}{(\mu+\phi)}$, which examines if the epidemic occurs. If $\mathcal{R}_0 < 1$, the Model (2) exhibits a notable characteristic where the disease-free equilibrium $\mathcal{E}_0 = (\mathcal{S}_0, 0, 0, 0) = \left(\dfrac{\lambda}{\mu}, 0, 0, 0\right)$ serves as a universal attractor within the invariant set Υ where,

$$\Upsilon = \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) : \mathcal{S} > 0, \quad \mathcal{E} > 0, \quad I > 0, \quad \mathcal{R} > 0, \quad \mathcal{S}+\mathcal{E}+I+\mathcal{R} \leq \frac{\lambda}{\mu}\right\}.$$

This implies that the disease will persist and sustain itself within the population. However, Model (2) has two equilibria, though, if $\mathcal{R}_0 > 1$. These are the endemic equilibrium $\overline{\mathcal{E}} = (\mathcal{S}^*, \mathcal{E}^*, I^*, \mathcal{R}^*)$ and the free of disease equilibrium $\mathcal{E}_0$. In this case, $\mathcal{E}_0$ is a source (or unstable), while $\overline{\mathcal{E}}$ acts as a global attractor within the interior of Υ provided $\theta = 0$.

However, in real-world scenarios, environmental noise consistently affects biological populations [3, 12], introducing variability that can significantly influence disease dynamics [22, 23]. Human-related epidemics tend to spread randomly as individuals come into contact with each other [24]. Therefore, stochastic models are often more suitable for describing such unpredictable dynamics. In the literature, random perturbations have been introduced into population models using both mathematical and biological perspectives [16, 9]. In this study, we follow the assumptions of Mao et al. [23], where model parameters fluctuate around their mean due to continuous environmental fluctuations. The model incorporates noise as a Brownian motion process, with intensity proportional to key epidemic variables within the stochastic SEIRS framework. Several

authors have emphasized that this approach is biologically sound and methodologically appropriate [28, 26]. This specification can be used as a Markov process for a small amounts of Δt in the model $\mathscr{X} = (\mathcal{S}; \mathcal{E}; I; \mathcal{R}; \mathcal{S})^{\mathscr{T}}$ [8];

$$\begin{aligned}
\mathscr{E}\left[\mathcal{S}(t+\Delta t) - \mathcal{S}(t) | \mathscr{X} = \chi\right] &\approx \left[\lambda - \frac{\beta_1 \mathcal{S} I}{1+\alpha I} - \beta_2 \mathcal{S}\mathcal{E} - \mu\mathcal{S} + \delta\mathcal{R}\right]\Delta t,\\
\mathscr{E}[\mathcal{E}(t+\Delta t) - \mathcal{E}(t) | \mathscr{X} = \chi] &\approx \left[\frac{\beta_1 \mathcal{S} I}{1+\alpha I} + \beta_2 \mathcal{S}\mathcal{E} - (\phi+\mu)\mathcal{E}\right]\Delta t,\\
\mathscr{E}[I(t+\Delta t) - I(t) | \mathscr{X} = \chi] &\approx [\phi\mathcal{E} - (\pi+\mu+\theta)I]\,\Delta t,\\
\mathscr{E}[\mathcal{R}(t+\Delta t) - \mathcal{R}(t) | \mathscr{X} = \chi] &\approx [\pi I - (\mu+\delta)\mathcal{R}]\,\Delta t,
\end{aligned}$$

and

$$\begin{aligned}
Var[\mathcal{S}(t+\Delta t) - \mathcal{S}(t) | \mathscr{X} = \chi] &\approx \varrho_1^2 \mathcal{S}^2 \Delta(t),\\
Var[\mathcal{E}(t+\Delta t) - \mathcal{E}(t) | \mathscr{X} = \chi] &\approx \varrho_2^2 \mathcal{E}^2 \Delta(t),\\
Var[I(t+\Delta t) - I(t) | \mathscr{X} = \chi] &\approx \varrho_3^2 I^2 \Delta(t),\\
Var[\mathcal{R}(t+\Delta t) - \mathcal{R}(t) | \mathscr{X} = \chi] &\approx \varrho_4^2 \mathcal{R}^2 \Delta(t).
\end{aligned}$$

Formally, this corresponds to the Model (2), by assuming the below stochastic system;

$$\begin{aligned}
d\mathcal{S} &= \left[\lambda - \frac{\beta_1 \mathcal{S} I}{1+\alpha I} - \beta_2 \mathcal{S}\mathcal{E} - \mu\mathcal{S} + \delta\mathcal{R}\right] dt + \varrho_1 \mathcal{S} d\mathscr{B}_1(t),\\
d\mathcal{E} &= \left[\frac{\beta_1 \mathcal{S} I}{1+\alpha I} + \beta_2 \mathcal{S}\mathcal{E} - (\phi+\mu)\mathcal{E}\right] dt + \varrho_2 \mathcal{E} d\mathscr{B}_2(t),\\
dI &= [\phi\mathcal{E} - (\pi+\mu+\theta)I]\, dt + \varrho_3 I d\mathscr{B}_3(t),\\
d\mathcal{R} &= [\pi I - (\mu+\delta)\mathcal{R}]\, dt + \varrho_4 \mathcal{R} d\mathscr{B}_4(t),
\end{aligned} \tag{3}$$

The given equation involves independent conventional one-dimensional Brownian motions, represented by $\mathscr{B}_i(t)$. The parameters ϱ_i^2 denote the intensities of the white noise, with i ranging from 1 to 4. The remainder of the parameters are identical to those in Model (2). In deterministic epidemic models, the basic rate of reproduction and the endemic equilibrium are two important components. These elements play a crucial role in understanding the dynamics of epidemics. The basic reproduction number represents the threshold at which an epidemic starts to spread, while the endemic equilibrium indicates the stable level of the disease within a population. However, when environmental noise is considered, the threshold for the spread of an epidemic in stochastic models becomes more complex to determine, and most stochastic models lack a clear endemic equilibrium.

As a result, stochastic models have been extensively studied for their stationary distributions. The ergodicity of these stationary distributions indicates that infectious diseases can persist over long periods from a bio-mathematical perspective, potentially leading to further developments in the disease's behavior. This study, therefore, focuses on two main objectives:

(i) A stochastic SEIRS epidemic model can be created by using a generalized incidence function $g(I)S$. By leveraging the characteristics of $g(I)$, we can establish the existence and uniqueness of a global positive solution.

(ii) The aim is to determine the disease's extinction threshold and analyze the stationary distribution of the stochastic epidemic model by incorporating the generalized incidence function.

By achieving these objectives, the study aims to enhance our understanding of the long-term behavior and control of infectious diseases in the presence of environmental noise, providing valuable insights for epidemic management and intervention strategies.

The paper is organized in a concise and academic manner. In Section 3, we provide preliminary results that will be utilized for our analysis. As stated in Section 4, a comprehensive global positive solution for the Model (3) is provided. Section 5 demonstrates that Model (3) displays an ergodic stationary distribution under specific conditions. In Section 6, we provide a concise overview of the necessary conditions for the disease to be eradicated. Section 7 presents numerical simulations that support the theoretical findings. In Section 8, we provide a concise overview of the key findings.

3 Preliminaries

In this paper, we utilize a concise and academic approach by considering $(\Omega, \mathscr{F}, \mathcal{P})$ as an entire probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ that meets the standard requirements. Specifically, the filtration is both increasing and right-continuous, and $\mathcal{F}_0$ involves all $\mathcal{P}$-null sets. Furthermore, on this probability space, $\mathscr{B}_i(t)$ can be defined for $(i = 1, 2, 3, 4)$.

Let, $\mathcal{R}_+^d = \left\{\chi \in \mathscr{R}^d : \chi_i > 0, 1 \leq i \leq d\right\}$.The standard equation for a d-dimensional Stochastic Differential Equation (SDE) is expressed as follows,

$$d\chi(t) = f(\chi(t), t)dt + g(\chi(t), t)d\mathscr{B}(t), \quad \text{for} \quad t \geq t_0, \tag{4}$$

with the starting value $\chi(0) = \chi_0 \in \mathscr{R}^d$. The n-dimensional Brownian motion $\mathscr{B}_i(t)$ is specified over the complete probability space $\left(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathcal{P}\right)$. Let $\mathcal{C}^{2,1}\left(\mathscr{R}^d \times [t_0, \infty]; \mathscr{R}_+\right)$ denotes the group of all non-negative functions $\mathcal{V}(\chi, t)$ defined on $\mathscr{R}_d \times [t_0, \infty]$ that are twice continuously differentiable with respect to χ and once differentiable with respect to $'t'$. As follows are the differential operators $\mathscr{L}$ of Model (2) [22],

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{\ell=1}^{d} f_\ell(\chi, t)\frac{\partial}{\partial \chi_\ell} + \frac{1}{2}\sum_{\ell,m=1}^{d}\left[g^{\mathscr{T}}(\chi, t)g(\chi, t)\right]_{\ell m}\frac{\partial^2}{\partial \chi_\ell \partial \chi_m}. \tag{5}$$

Acting $\mathscr{L}$ stands for a function $\mathcal{V} \in \mathcal{C}^{2,1}\left(\mathscr{R}^d \times [t_0, \infty]; \mathscr{R}_+\right)$, we have,

$$\mathscr{L}\mathcal{V}(\chi, t) = \mathcal{V}_t(\chi, t) + \mathcal{V}_\chi(\chi, t)f(\chi, t) + \frac{1}{2}trace\left[\mathcal{V}_{\chi\chi}(\chi, t)g^{\mathscr{T}}(\chi, t)g(\chi, t)\right],$$

where

$$\mathcal{V}_t = \frac{\partial \mathcal{V}}{\partial t}, \qquad \mathcal{V}_\chi = \left(\frac{\partial \mathcal{V}}{\partial \chi_1}, \frac{\partial \mathcal{V}}{\partial \chi_2}, \ldots, \frac{\partial \mathcal{V}}{\partial \chi_d}\right), \qquad \mathcal{V}_{\chi\chi} = \left(\frac{\partial^2 \mathcal{V}}{\partial \chi_i \partial \chi_j}\right)_{d\times d}.$$

Thus, by Ito's formula, if $\chi(t) \in \mathscr{R}^d$, then,

$$d\mathcal{V}(\chi(t), t) = \mathscr{L}\mathcal{V}(\chi(t), t)dt + \mathcal{V}_\chi(\chi(t), t)g(\chi(t), t)d\mathscr{B}(t).$$

The theory that follows will then focus on the stationary distribution [11], we offer some definitions and known facts in the section that follows. Suppose we have a homogeneous Markov

procedure $\mathscr{X}(t)$ within d-dimensional Euclidean space $\mathscr{E}_d$, and we describe it using the items that follow SDE,

$$d\mathscr{X}(t) = b(\mathscr{X})dt + \sum_{q=1}^{l} g_q(\mathscr{X})d\mathscr{B}_q(t). \tag{6}$$

In the diffusion matrix corresponding to (6) have the form,

$$\mathscr{A}(\chi) = (b_{ij}(\chi)), \quad b_{ij}(\chi) = \sum_{q=1}^{l} g_q^i(\chi)g_q^j(\chi). \tag{7}$$

Lemma 3.1. [11] *Consider a bounded region $\mathscr{D}$ contained within $\mathscr{E}_d$, with a smooth boundary denoted by Υ. It possesses the following features:*

- $\mathscr{H}_1$ *: There is a positive essentially unchanged constant $\mathscr{M} > 0$, such that every value of $\chi \in \mathscr{D}$ and $\gamma \in \mathscr{R}^d$, the inequality $\sum_{i,j=1}^{l} a_{ij}(\chi)\gamma_i\gamma_j \geq \mathscr{M}|\gamma|^2$ holds, where $\mathscr{M}$ is a constant with a positive value.*
- $\mathscr{H}_2$ *: There exists a $\mathcal{C}^2$-function $\mathscr{V} \geq 0$, such that $\mathscr{L}\mathscr{V} < 0$ for all points in $\mathscr{E}_d \backslash \mathscr{D}$. Therefore, the Markov process $\mathscr{X}(t)$ exhibits a distinctive stationary ergodic distribution denoted by $\tilde{\omega}(.)$.*

To be specific,

$$\mathscr{P}_\chi\left\{\lim_{\mathscr{T}\to\infty}\frac{1}{\mathscr{T}}\int_0^{\mathscr{T}} f(\mathscr{X}(t))dt = \int_{\mathscr{E}_d} f(\chi)\tilde{\omega}(d\chi)\right\} = 1, \quad \forall \quad \chi \in \mathscr{E}_d.$$

In this case, $f(\cdot)$ corresponds to a function integral with a measure $\tilde{\omega}$.

4 Uniqueness and Existence of Positive Global Solutions

When assessing the dynamic behavior of the epidemic model, the primary factor to consider is whether a broadly effective cure is accessible. The analysis that follows looks into the presence and singularity of the global positive solution in order to explore Model (2)'s long-term behavior.

Theorem 4.1. *For a given starting value, $\mathscr{X}(0) = (\mathcal{S}(0), \mathcal{E}(0), I(0), \mathcal{R}(0)) \in \mathscr{R}_+^4$, a positive and unique solution can be obtained. The process $\mathscr{X}(t)$ consists of four components: $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$. In the order system (3), it is defined as a solution in $\mathscr{R}_+^4$ with probability 1 for $t \geq 0$. The solution $(\mathcal{S}(t), \mathcal{E}(t), I(t), \mathcal{R}(t)) \in \mathscr{R}_+^4 \; \forall \; t \geq 0$ is guaranteed almost surely (a.s.).*

Proof. Given the model's coefficients, it can be observed that they satisfy the local Lipschitz condition. This implies that for any starting value $(\mathcal{S}(0), \mathcal{E}(0), I(0), \mathcal{R}(0)) \in \mathscr{R}_+^4$, there exists a locally distinct solution $(\mathcal{S}(t), \mathcal{E}(t), I(t), \mathcal{R}(t))$. $'t'$ is defined within the interval $[0, \tau_d)$, with τ_d denoting the explosion time [22]. In order to establish the global nature of the solution, it is necessary to demonstrate that $\tau_d = +\infty$ a.s. Assume that k_0 is a sufficiently large value, such that the beginning values of $\mathcal{S}(0)$, $\mathcal{E}(0)$, $I(0)$, and $\mathcal{R}(0)$ all collapse within the interval $[1/k_0, k_0]$. The halting time can be represented using the following sequence for each and every integer $k \geq k_0$.

$$\tau_k = \inf\left\{t \in [0, \tau_d) : \min\left\{\mathcal{S}(t), \mathcal{E}(t), I(t), \mathcal{R}(t)\right\} \leq \frac{1}{k}, \quad \text{or} \quad \max\left\{\mathcal{S}(t), \mathcal{E}(t), I(t), \mathcal{R}(t)\right\} \geq k\right\}.$$

Throughout this paper, we adopt the convention that $\inf \emptyset = \infty$ (in general, $\emptyset$ is as the null set). Clearly, as the value of $\mathcal{k}$ approaches infinity, it is apparent that $\tau_{\mathcal{k}}$ is increasing. Consider the limit of $\tau_{\mathcal{k}}$ as $\mathcal{k}$ approaches infinity, denoted by τ_∞. Therefore, it follows that τ_∞ is less than or equal to τ_z a.s. In this case, it can be shown that τ_z is a.s. infinite, and for all $t \geq 0$, the variables $(\mathcal{S}(t), \mathcal{E}(t), I(t), \mathcal{R}(t))$ belong to $\mathscr{R}_+^4$ a.s. The only thing left to do is verify $\tau_\infty = +\infty$ a.s. In case the proposition is not followed, there exists a constant $\mathscr{T} > 0$ and a $\phi \in (0,1)$ respectively, such that,

$$\mathscr{P}\left\{\tau_{\mathcal{k}} \leq \mathscr{T}\right\} \geq \phi, \quad \forall \quad \mathcal{k} \geq \mathcal{k}_1.$$

Define a non-negative $\mathcal{C}^2$ function $\mathcal{V} : \mathscr{R}_+^4 \to \mathscr{R}_+$ by,

$$\mathcal{V}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) = (\mathcal{S} - 1 - \ln \mathcal{S}) + (\mathcal{E} - 1 - \ln \mathcal{E}) + (I - 1 - \ln I) + (\mathcal{R} - 1 - \ln \mathcal{R}).$$

Applying Ito's formula, it will be,

$$\begin{aligned} d\mathcal{V}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) = \mathscr{L}\mathcal{V}(\mathcal{S}, \mathcal{E}, I, \mathcal{R})\, dt + \varrho_1(\mathcal{S} - 1)d\mathscr{B}_1(t) + \varrho_2(\mathcal{E} - 1)d\mathscr{B}_2(t) \\ + \varrho_3(I - 1)d\mathscr{B}_3(t) + \varrho_4(\mathcal{R} - 1)d\mathscr{B}_4(t), \end{aligned}$$

therefore,

$$\begin{aligned} \mathscr{L}\mathcal{V} &= \left(1 - \frac{1}{\mathcal{S}}\right)\left(\lambda - \frac{\beta_1 \mathcal{S} I}{1 + \alpha I} - \beta_2 \mathcal{S}\mathcal{E} - \mu\mathcal{S} + \delta\mathcal{R}\right) + \left(1 - \frac{1}{\mathcal{E}}\right)\left(\frac{\beta_1 \mathcal{S} I}{1 + \alpha I} + \beta_2 \mathcal{S}\mathcal{E} - (\phi + \mu)\mathcal{E}\right) \\ &\quad + \left(1 - \frac{1}{I}\right)(\phi\mathcal{E} - (\mu + \theta + \pi)I) + \left(1 - \frac{1}{\mathcal{R}}\right)(\pi I - (\mu + \delta)\mathcal{R}) + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2}{2} \\ &= \lambda + 4\mu + \phi + \delta + \pi - \mu(\mathcal{S} + \mathcal{E} + I + \mathcal{R}) + \frac{\beta_1 I \mathcal{S}^2}{1 + \alpha I} + \frac{\beta_1 I \mathcal{S}\mathcal{E}}{1 + \alpha I} + \beta_2(\mathcal{S} + \mathcal{E}) - \frac{\phi\mathcal{E}}{I} - \frac{\lambda}{\mathcal{S}} \\ &\quad - \frac{\delta\mathcal{R}}{\mathcal{S}} - \frac{\pi I}{\mathcal{R}} + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2}{2} \\ &< \lambda + 4\mu + \phi + \delta + \pi + \beta_1 + \beta_2 + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_3^2 + \varrho_4^2}{2} \\ &< \mathcal{K}_0, \end{aligned}$$

where $\mathcal{K}_0$ is an appropriate constant that does not depend on $\mathcal{S}(t), \mathcal{E}(t), I(t)$ and $\mathcal{R}(t)$. Consequently,

$$\begin{aligned} d\mathcal{V}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) = \mathcal{K}_0\, dt + \varrho_1(\mathcal{S} - 1)d\mathscr{B}_1(t) + \varrho_2(\mathcal{E} - 1)d\mathscr{B}_2(t) + \varrho_3(I - 1)d\mathscr{B}_3(t) \\ + \varrho_4(\mathcal{R} - 1)d\mathscr{B}_4(t). \end{aligned}$$

Take the above equation and integrate both sides from 0 to $\tau_{\mathcal{k}} \wedge \mathscr{T}$ for every $\mathcal{k} \geq \mathcal{k}_0$. The remaining part of the proof for Theorem 4.1 follows a similar approach to that in Mao et al. [23] and is omitted here. □

5 Ergodicity and Stationary Distribution Analysis

In an epidemic model, it's imperative to consider when the disease becomes widespread in the community first. The issue needs to be addressed by demonstrating that the equilibrium in the relevant deterministic model is either globally stable or a globally attractive force. However, Model (3) does not demonstrate endemic equilibrium. In this part, Khasminskii [11] theory implies that sickness will continue, so we validate the existence of an ergodic stationary distribution.

Define a parameter,

$$\mathcal{R}_{\mathfrak{0}}^{S} = \frac{(\beta_1\phi\lambda + \beta_2)}{\left(\mu + \frac{\varrho_1^2}{2}\right)\left(\phi + \mu + \frac{\varrho_2^2}{2}\right)\left(\pi + \theta + \mu + \frac{\varrho_3^2}{2}\right)}.$$

Theorem 5.1. *Make the assumption $\mathcal{R}_{\mathfrak{0}}^{S} > 1$. For instance, let's take any beginning value $(\mathcal{S}(0), \mathcal{E}(0), I(0), \mathcal{R}(0)) \in \mathscr{R}_+^4$ for the Model (3). The ergodic property is exhibited by this model, which has a unique stationary distribution $\tilde{\omega}(\cdot)$.*

Proof. From (7), it is evident that the diffusion matrix of Model (3) is of the form,

$$\mathscr{A} = \begin{pmatrix} \varrho_1^2\mathcal{S}^2 & 0 & 0 & 0 \\ 0 & \varrho_2^2\mathcal{E}^2 & 0 & 0 \\ 0 & 0 & \varrho_3^2 I^2 & 0 \\ 0 & 0 & 0 & \varrho_4^2\mathcal{R}^2 \end{pmatrix}.$$

As per Lemma 3.1, we need to examine both the terms and conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ respectively. Choose $\mathcal{M} = \min_{(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\overline{\mathcal{D}}_\varrho \subset \mathscr{R}_+^4} \left\{\varrho_1^2\mathcal{S}^2, \varrho_2^2\mathcal{E}^2, \varrho_3^2 I^2, \varrho_4^2\mathcal{R}^2\right\}$, one may have that,

$$\sum_{n,m=1}^{4} a_{nm}(\mathcal{S},\mathcal{E},I,\mathcal{R})\gamma_n\gamma_m = \varrho_1^2\mathcal{S}^2\gamma_1^2 + \varrho_2^2\mathcal{E}^2\gamma_2^2 + \varrho_3^2 I^2\gamma_3^2 + \varrho_4^2\mathcal{R}^2\gamma_4^2 \geq \mathcal{M}\left|\gamma\right|^2,$$

$(\mathcal{S},\mathcal{E},I,\mathcal{R}) \in \overline{\mathcal{D}}_\varrho$, $\gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4) \in \mathscr{R}_+^4$. In order to validate the condition $(\mathcal{H}_1)$ in Lemma 3.1. As a $\mathcal{C}^2$-function, construct $\mathscr{N} : \mathscr{R}_+^4 \to \mathscr{R}$ as follows:

$$\begin{aligned}\mathscr{N}(\mathcal{S},\mathcal{E},I,\mathcal{R}) &= \mathcal{M}(\mathcal{S}+\mathcal{E}+I+\mathcal{R} - m_1\ln\mathcal{S} - m_2\ln\mathcal{E} - m_3\ln I) + \frac{1}{\vartheta+1}(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta+1} \\ &\quad - \ln\mathcal{S} - \ln\mathcal{E} - \ln\mathcal{R} + (\mathcal{S}+\mathcal{E}+I+\mathcal{R}) \\ &= \mathcal{M}\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5 + \mathcal{V}_6,\end{aligned}$$

where ϑ is a constant that meets the specified condition, $0 < \vartheta < \dfrac{2\mu}{\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2}$,

$$\begin{aligned}&\mathcal{V}_1 = \mathcal{S}+\mathcal{E}+I+\mathcal{R} - m_1\ln\mathcal{S} - m_2\ln\mathcal{E} - m_3\ln I, \\ &\mathcal{V}_2 = \frac{1}{\vartheta+1}(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta+1}, \qquad \mathcal{V}_3 = -\ln\mathcal{S}, \qquad \mathcal{V}_4 = -\ln\mathcal{E}, \\ &\mathcal{V}_5 = -\ln\mathcal{R}, \qquad \mathcal{V}_6 = \mathcal{S}+\mathcal{E}+I+\mathcal{R}, \\ &m_1 = \frac{\lambda}{\mu + \frac{\varrho_1^2}{2}}, \qquad m_2 = \frac{\lambda}{\phi + \mu + \frac{\varrho_2^2}{2}}, \qquad m_3 = \frac{\lambda}{\pi + \theta + \mu + \frac{\varrho_3^2}{2}},\end{aligned}$$

then, $\mathcal{M} > 0$ satisfies the given condition,

$$-\mathcal{M}\lambda + \mathcal{C} \leq -2. \tag{8}$$

Therefore, it is clear to overlook,

$$\lim \inf_{\mathcal{k}\to\infty, (\mathcal{S},\mathcal{E},I,\mathcal{R})\in\mathscr{R}_+^4\setminus\mathcal{U}_\mathcal{k}} \mathscr{N}(\mathcal{S},\mathcal{E},I,\mathcal{R}) = +\infty,$$

where $\mathcal{U}_{\kappa} = \left(\frac{1}{\kappa}, \kappa\right) \times \left(\frac{1}{\kappa}, \kappa\right) \times \left(\frac{1}{\kappa}, \kappa\right) \times \left(\frac{1}{\kappa}, \kappa\right)$. Moreover, $\mathcal{N}(\mathcal{S}, \mathcal{E}, I, \mathcal{R})$ is a continuous function. Therefore, $\mathcal{N}(\mathcal{S}, \mathcal{E}, I, \mathcal{R})$ should have a minimum point $(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{E}}_0, \widetilde{I}_0, \widetilde{\mathcal{R}}_0)$ in the interior of $\mathcal{R}_+^4$. Then, we describe a non-negative $\mathcal{C}^2$-function $\widetilde{\mathcal{V}} : \mathscr{R}_+^4 \to \mathscr{R}_+$ in the following manner,

$$\widetilde{\mathcal{V}}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) = \mathcal{N}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) - \mathcal{N}(\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{E}}_0, \widetilde{I}_0, \widetilde{\mathcal{R}}_0).$$

By applying Ito's formula, we have,

$$\begin{aligned}
\mathscr{L}\mathcal{V}_1 &= \lambda - \frac{\beta_1 \mathcal{S} I}{1+\alpha I} - \beta_2 \mathcal{S}\mathcal{E} - \mu\mathcal{S} + \delta\mathcal{R} + \frac{\beta_1 \mathcal{S} I}{\mathcal{E}(1+\alpha I)} - \beta_2\mathcal{S} - (\mu+\phi) - \frac{\phi\mathcal{E}}{I} \\
&\quad - (\pi+\mu+\theta) - m_1\frac{\lambda}{\mathcal{S}} - m_1\frac{\beta_1 I}{1+\alpha I} - m_1\beta_2\mathcal{E} + m_1\mu + m_1\frac{\delta\mathcal{R}}{\mathcal{S}} + m_1\frac{\varrho_1^2}{2} \\
&\quad - m_2\frac{\beta_1 \mathcal{S} I}{\mathcal{E}(1+\alpha I)} - m_2\beta_2\mathcal{S} - m_2(\mu+\phi) + m_2\frac{\varrho_2^2}{2} - m_3\frac{\phi\mathcal{E}}{I} - m_3(\pi+\mu+\theta) + m_3\frac{\varrho_3^2}{2} \\
&= \lambda - m_1\frac{\lambda}{\mathcal{S}} - m_2\frac{\beta_1 \mathcal{S} I}{\mathcal{E}(1+\alpha I)} - m_3\frac{\phi\mathcal{E}}{I} - m_1\frac{\beta_1 I}{1+\alpha I} + m_2(\beta_2\mathcal{E} + \beta_2\mathcal{S}) + m_1\left(\mu + \frac{\varrho_1^2}{2}\right) \\
&\quad + m_2\left(\mu+\phi+\frac{\varrho_2^2}{2}\right) + m_3\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right).
\end{aligned}$$

Applying the inequality relation $a + b \geq 2\sqrt{ab}$ holds true when both a and b are positive.

$$\begin{aligned}
\mathscr{L}\mathcal{V}_1 &\leq -2\left(\frac{m_2\beta_1\beta_2\mathcal{S} I}{\mathcal{E}(\alpha I)}\right)^{\frac{1}{2}} - 2\left(\frac{m_1 m_3 \lambda\phi\mathcal{E}}{\mathcal{S} I}\right)^2 + m_1\frac{\beta_1 I}{1+\alpha I} + \lambda + m_1\left(\mu+\frac{\varrho_1^2}{2}\right) \\
&\quad + m_2\left(\mu+\phi+\frac{\varrho_2^2}{2}\right) + m_3\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right) \\
&\leq -4\left(m_1 m_2 m_3\beta_1\beta_2\lambda\phi\right)^{\frac{1}{4}} + m_1\frac{\beta_1 I}{1+\alpha I} + m_2\beta_2\mathcal{S} + \lambda + m_1\left(\mu+\frac{\varrho_1^2}{2}\right) \\
&\quad + m_2\left(\mu+\phi+\frac{\varrho_2^2}{2}\right) + m_3\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right) \\
&= -4\lambda\left\{\left(\frac{\beta_1\beta_2\phi}{\left(\mu+\frac{\varrho_1^2}{2}\right)\left(\mu+\phi+\frac{\varrho_2^2}{2}\right)\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right)}\right)^{\frac{1}{4}} - 1\right\} + m_1\frac{\beta_1 I}{1+\alpha I} + m_2\beta_2\mathcal{S}.
\end{aligned} \tag{9}$$

Similarly, we get,

$$\begin{aligned}
\mathscr{L}\mathcal{V}_2 &= (\lambda - \mu\mathcal{S} - \mu\mathcal{E} - (\mu+\theta)I - \mu\mathcal{R})(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta} \\
&\quad + \frac{\vartheta}{2}(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta-1}\left(\varrho_1^2\mathcal{S}^2 + \varrho_2^2\mathcal{E}^2 + \varrho_3^2 I^2 + \varrho_4^2\mathcal{R}^2\right) \\
&\leq (\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta}\left[\lambda - \mu(\mathcal{S}+\mathcal{E}+I+\mathcal{R})\right] \\
&\quad + \frac{\vartheta}{2}(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta+1}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2) \\
&= \lambda(\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{\vartheta} - \left[\mu - \frac{\vartheta}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{1+\vartheta} \\
&\leq \mathscr{B} - \frac{1}{2}\left[\mu - \frac{\vartheta}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](\mathcal{S}+\mathcal{E}+I+\mathcal{R})^{1+\vartheta} \\
&\leq \mathscr{B} - \frac{1}{2}\left[\mu - \frac{\vartheta}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](\mathcal{S}^{1+\vartheta} + \mathcal{E}^{1+\vartheta} + I^{1+\vartheta} + \mathcal{R}^{1+\vartheta}).
\end{aligned} \tag{10}$$

We can also get,

$$\mathscr{L}\mathcal{V}_3 = -\frac{\lambda}{\mathcal{S}} + \frac{\beta_1}{\alpha} + \beta_2\mathcal{E} + \frac{\delta\mathcal{R}}{\mathcal{S}} + \mu + \frac{\varrho_1^2}{2}, \tag{11}$$

$$\mathscr{L}\mathcal{V}_4 = -\frac{\beta_1\mathcal{S}I}{\mathcal{E}(1+\alpha I)} - \beta_2\mathcal{S} + \mu + \phi + \frac{\varrho_2^2}{2}, \tag{12}$$

$$\mathscr{L}\mathcal{V}_5 = -\frac{\pi I}{\mathcal{R}} + \mu + \delta + \frac{\varrho_4^2}{2}, \tag{13}$$

$$\mathscr{L}\mathcal{V}_6 = \lambda - \mu(\mathcal{S} + \mathcal{E} + I + \mathcal{R}). \tag{14}$$

We obtained,

$$\begin{aligned}
\mathscr{L}\mathcal{V} = &-\mathcal{M}\lambda + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \mathcal{M}m_2\beta_2\mathcal{S} - \frac{1}{2}\left[\mu - \frac{\vartheta}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right] \\
&\times (\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) - \frac{\lambda}{\mathcal{S}} + \frac{\beta_1 I}{1+\alpha} + \beta_2\mathcal{E} + \frac{\delta\mathcal{R}}{\mathcal{S}} + \mu + \frac{\varrho_1^2}{2} - \frac{\beta_1\mathcal{S}I}{\mathcal{E}(1+\alpha I)} \\
&- \beta_2\mathcal{S} + \mu + \phi + \frac{\varrho_2^2}{2} - \frac{\pi I}{\mathcal{R}} + \mu + \delta + \frac{\varrho_4^2}{2} - \lambda - \mu(\mathcal{S} + \mathcal{E} + I + \mathcal{R}) \\
\leq &-\mathcal{M}\lambda + \frac{\mathcal{M}m - 1\beta_1 I}{1+\alpha I} + \mathcal{M}m_2\beta_2\mathcal{S} - \frac{\beta_1\mathcal{S}I}{\mathcal{E}(1+\alpha I)} + \frac{\beta_1 I}{1+\alpha} \\
&- \frac{1}{2}\left[\mu - \frac{\vartheta}{2}(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) \\
&- \frac{\lambda}{\mathcal{S}} - \frac{\pi I}{\mathcal{R}} - \mu(\mathcal{S} + \mathcal{E} + I + \mathcal{R}) + \lambda + 3\mu + \delta + \phi + \beta_2\mathcal{S} + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2}.
\end{aligned}$$

At this point, we construct a compact subset of Θ that satisfies condition $\mathcal{H}_2$ as stated in Lemma 3.1. This bounded and closed set is defined as follows:

$$\Theta = \left\{\xi_1 \leq \mathcal{S} \leq \frac{1}{\xi_1}, \quad \xi_2 \leq I \leq \frac{1}{\xi_2}, \quad \xi_1 \leq \mathcal{E} \leq \frac{1}{\xi_3}, \quad \xi_1 \leq \mathcal{R} \leq \frac{1}{\xi_4}\right\},$$

In cases where $\xi_i > 0$ for constants $(i = 1, 2, 3, 4)$, certain conditions must be satisfied:

$$-\frac{\lambda}{\xi_1} + \mathcal{H} \leq -1, \tag{15}$$

$$-\mathcal{M}\lambda - \mathcal{M}m_1\beta_1\xi_1 + \mathcal{M}m_2\beta_2\xi_1 + \mathcal{K} \leq -1, \tag{16}$$

$$-2\left(\frac{\beta_1\mathcal{M}}{\xi_1}\right)^{\frac{1}{2}} + \beta_2\xi_1 + \mathcal{H} \leq -1, \tag{17}$$

$$-\frac{\pi}{\xi_1} + \mathcal{H} \leq -1, \tag{18}$$

$$-\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{(\vartheta+1)}} + \mathcal{G} \leq -1, \tag{19}$$

$$-\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{2(\vartheta+1)}} + \mathcal{F} \leq -1, \tag{20}$$

$$-\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{4(\vartheta+1)}} + \mathcal{J} \leq -1, \tag{21}$$

$$-\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{3(\vartheta+1)}} + \mathcal{Y} \leq -1. \tag{22}$$

It is shown that $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{J}$, and $\mathcal{Y}$ are positive constants which is given explicitly in the expression (15)−(22). For our convenience, this set $\mathscr{R}^4_+ \backslash \Theta$ is split into 8 domains, so that we can calculate it more easily,

$$\begin{aligned}
\Theta_1 &= \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : 0 < \mathcal{S} < \xi_1\right\}, \\
\Theta_2 &= \left\{(S, E, I, R) \in \mathscr{R}^4_+ : 0 < I < \xi_2, \quad \mathcal{S} \geq \xi_1\right\}, \\
\Theta_3 &= \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : \mathcal{S} \geq \xi_1 I \geq \xi_2, \quad 0 < \mathcal{E} < \xi_3\right\}, \\
\Theta_4 &= \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : 0 < \mathcal{R} < \xi_4, \quad I \geq \xi_2\right\}, \\
\Theta_5 &= \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : \mathcal{S} > \frac{1}{\xi_1}\right\}, \quad \Theta_6 = \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : I > \frac{1}{\xi_2}\right\}, \\
\Theta_7 &= \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : \mathcal{E} > \frac{1}{\xi_3}\right\}, \quad \Theta_8 = \left\{(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \mathscr{R}^4_+ : \mathcal{R} > \frac{1}{\xi_4}\right\}.
\end{aligned}$$

Next, we will show that the inequality $\mathscr{L}\mathcal{V}(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \leq -1$ holds on the set $\mathscr{R}^4_+ \backslash \Theta$. As a result, the problem can be solved in each of the eight domains listed above.

Case I: Suppose that $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_1$, one may get that,

$$\begin{aligned}
\mathscr{L}\mathcal{V} &\leq -\frac{\lambda}{\mathcal{S}} + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \mathcal{M}m_2\beta_2\mathcal{S} - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right] \\
&\quad \times (\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) + \lambda + 3\mu + \delta + \phi + \mathscr{B} + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2} \\
&\leq -\frac{\lambda}{\mathcal{S}} + \mathcal{H} \leq -\frac{\lambda}{\xi_1} + \mathcal{H},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\mathcal{H} = \sup_{(\mathcal{S},\mathcal{E},I,\mathcal{R}) \in \mathscr{R}^4_+} &\left\{\frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \mathcal{M}m_2\beta_2\mathcal{S} - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\right\} \\
&\times (\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) + \lambda + 3\mu + \delta + \phi + \mathscr{B} + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2}.
\end{aligned} \tag{24}$$

By using (15), we can infer this $\mathscr{L}\mathcal{V} \leq -1$ for all $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_1$.

Case II: Suppose that $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_2$, one can get that,

$$\begin{aligned}
\mathscr{L}\mathcal{V} &\leq -\mathcal{M}\lambda + \frac{\mathcal{M}m_1\beta_1 I\mathcal{S}}{1+\alpha I} + \beta_2\mathcal{S}\mathcal{M}m_2 - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right] \\
&\quad \times (\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) + \beta_2\mathcal{S} + 3\mu + \delta + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2} \\
&\leq -\mathcal{M}\lambda + \frac{\mathcal{M}m_1\beta_1\xi_2}{1+\alpha I} + \beta_2\xi_1\mathcal{M}m_2 + \mathcal{K} \\
&\leq -\mathcal{M}\lambda + \frac{\mathcal{M}m_1\beta_1\xi_2}{\alpha} + \beta_2\xi_1\mathcal{M}m_2 + \mathcal{K}.
\end{aligned}$$

Choosing $\xi_2 = \xi_1^2$ and $\alpha = \xi_1$,

$$\mathscr{L}\mathcal{V} \leq -\mathcal{M}\lambda + \mathcal{M}m_1\beta_1\xi_1 + \beta_2\xi_1\mathcal{M}m_2 + \mathcal{K}. \tag{25}$$

Because of (16), it can be inferred that for a very small ξ_1. If $\mathscr{L}\mathcal{V} \leq -1$, it applies to any $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_2$.

Case III: Suppose that $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_3$,

$$\begin{aligned}
\mathscr{L}\mathcal{V} &\leq -2\left(\frac{\mathcal{M}\beta_1 I\mathcal{S}}{1+\alpha I}\right)^{\frac{1}{2}} + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \mathcal{M}m_1\beta_2 I \\
&\quad - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) \\
&\quad + \beta_2\mathcal{S} + 3\mu + \delta + \phi + \lambda + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2} \\
&\leq -2\left(\frac{\mathcal{M}\beta_1 I\mathcal{S}}{\alpha}\right)^{\frac{1}{2}} + \beta_2 I + \mathcal{H} \\
&\leq -2\left(\frac{\mathcal{M}\beta_1\xi_1\xi_2}{\alpha}\right)^{\frac{1}{2}} + \beta_2\xi_2 + \mathcal{H} \\
&\leq -2\left(\frac{\beta_1\mathcal{M}}{\xi_1}\right)^{\frac{1}{2}} + \beta_2\xi_1 + \mathcal{H}. \qquad (26)
\end{aligned}$$

Considering the condition (17), we can obtain that $\mathscr{L}\mathcal{V} \leq -1$ for any $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_3$.

Case IV: Suppose that $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_4$, we have,

$$\begin{aligned}
\mathscr{L}\mathcal{V} &\leq -\frac{\pi I}{\mathcal{R}} + \frac{\mathcal{M}m_1\beta_1 I\mathcal{S}}{1+\alpha I} - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right] \\
&\quad \times (\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) + \beta_2\mathcal{M} + 3\mu + \delta + \phi + \lambda + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2} \\
&\leq -\frac{\pi I}{\mathcal{R}} + \mathcal{H} \\
&\leq -\frac{\pi}{\xi_1} + \mathcal{H}. \qquad (27)
\end{aligned}$$

By using (18), we can achieve that $\mathscr{L}\mathcal{V} \leq -1$ for all Θ_4.

Case V: Suppose that $(\mathcal{S}, \mathcal{E}, I, \mathcal{R}) \in \Theta_5$, we have,

$$\begin{aligned}
\mathscr{L}\mathcal{V} &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{S}^{\vartheta+1} - \frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{S}^{\vartheta+1} \\
&\quad - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right](E^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}) + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} \\
&\quad + \beta_2\mathcal{M} + \mathscr{B} + 3\mu + \delta + \phi + \lambda + \frac{\varrho_1^2 + \varrho_2^2 + \varrho_4^2}{2} \\
&\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{S}^{\vartheta+1} + \mathcal{G} \\
&\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{\vartheta+1}} + \mathcal{G}, \qquad (28)
\end{aligned}$$

where

$$\begin{aligned}\mathcal{G} = \sup_{(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\mathcal{R}_+^4} & -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\mathcal{S}^{\vartheta+1}\\ & -\frac{1}{2}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\left(\mathcal{E}^{\vartheta+1}+I^{\vartheta+1}+\mathcal{R}^{\vartheta+1}\right)\\ & +\frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I}+\beta_2\mathcal{M}+\mathscr{B}+3\mu+\delta+\phi+\lambda+\frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}.\end{aligned} \tag{29}$$

Together with (19), implies that $\mathscr{L}\mathcal{V}\leq -1$ on Θ_5.

Case VI: Suppose that $(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\Theta_6$, we have,

$$\begin{aligned}\mathscr{L}\mathcal{V} &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]I^{\vartheta+1}-\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]I^{\vartheta+1}\\ &\quad -\frac{1}{2}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1}+\mathcal{E}^{\vartheta+1}+\mathcal{R}^{\vartheta+1}\right)\\ &\quad +\frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I}+\beta_2\mathcal{M}+\mathscr{B}+3\mu+\delta+\phi+\lambda+\frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}\\ &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]I^{\vartheta+1}+\mathcal{J}\\ &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\frac{1}{\xi_1^{2(\vartheta+1)}}+\mathcal{J},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\mathcal{J} = \sup_{(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\mathcal{R}_+^4} & -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]I^{\vartheta+1}\\ & -\frac{1}{2}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1}+\mathcal{E}^{\vartheta+1}+\mathcal{R}^{\vartheta+1}\right)\\ & +\frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I}+\beta_2\mathcal{M}+\mathscr{B}+3\mu+\delta+\phi+\lambda+\frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}.\end{aligned}$$

According to inequality (20), it is possible to accomplish $\mathscr{L}\mathcal{V}\leq -1$ on Θ_6.

Case VII: Suppose that $(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\Theta_7$, we have,

$$\begin{aligned}\mathscr{L}\mathcal{V} &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\mathcal{E}^{\vartheta+1}-\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\mathcal{E}^{\vartheta+1}\\ &\quad -\frac{1}{2}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1}+I^{\vartheta+1}+\mathcal{R}^{\vartheta+1}\right)\\ &\quad +\frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I}+\beta_2\mathcal{M}+\mathscr{B}+3\mu+\delta+\phi+\lambda+\frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}\\ &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\mathcal{E}^{\vartheta+1}+\mathcal{K}\\ &\leq -\frac{1}{4}\left[\mu-\frac{1}{2}\vartheta(\varrho_1^2\vee\varrho_2^2\vee\varrho_3^2\vee\varrho_4^2)\right]\frac{1}{\xi_1^{4(\vartheta+1)}}+\mathcal{K},\end{aligned} \tag{31}$$

where

$$\mathcal{K} = \sup_{(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\mathcal{R}_+^4} -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{E}^{\vartheta+1} - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1} + I^{\vartheta+1} + \mathcal{R}^{\vartheta+1}\right) + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \beta_2\mathcal{M} + \mathscr{B} + 3\mu + \delta + \phi + \lambda + \frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}.$$

This result, in conjunction with (21), indicates that $\mathscr{L}\mathcal{V} \leq -1$ for all Θ_7.

Case VIII: Suppose that $(\mathcal{S},\mathcal{E},I,\mathcal{R}) \in \Theta_8$, we have,

$$\begin{aligned}\mathscr{L}\mathcal{V} &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{R}^{\vartheta+1} - \frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{R}^{\vartheta+1} \\ &\quad - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1}\right) \\ &\quad + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \beta_2\mathcal{M} + \mathscr{B} + 3\mu + \delta + \phi + \Lambda + \frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2} \\ &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{R}^{\vartheta+1} + \mathcal{Y} \\ &\leq -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\frac{1}{\xi_1^{3(\vartheta+1)}} + \mathcal{Y}, \end{aligned} \tag{32}$$

where

$$\mathcal{Y} = \sup_{(\mathcal{S},\mathcal{E},I,\mathcal{R})\in\mathcal{R}_+^4} -\frac{1}{4}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\mathcal{R}^{\vartheta+1} - \frac{1}{2}\left[\mu - \frac{1}{2}\vartheta(\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2)\right]\left(\mathcal{S}^{\vartheta+1} + \mathcal{E}^{\vartheta+1} + I^{\vartheta+1}\right) + \frac{\mathcal{M}m_1\beta_1 I}{1+\alpha I} + \beta_2\mathcal{M} + \mathscr{B} + 3\mu + \delta + \phi + \lambda + \frac{\varrho_1^2+\varrho_2^2+\varrho_4^2}{2}.$$

Applying (22), we get $\mathscr{L}\mathcal{V} \leq -1$ on Θ_8.

It follows from (23), (25)−(28) and (30)−(32) that the condition ξ_1 holds, provided that E is sufficiently small,

$$\mathscr{L}\mathcal{V}(\mathcal{S},\mathcal{E},I,\mathcal{R}) \leq -1, \quad \forall \quad (\mathcal{S},\mathcal{E},I,\mathcal{R}) \in \mathcal{R}_+^4 \backslash \Theta.$$

It is evident that Lemma 3.1, satisfies the condition $(\mathcal{H}_2)$. This Model (3) possesses a stationary distribution and exhibits ergodic behavior. And with that, the proof is complete. □

Remark 5.1. *Theorem 5.1 demonstrates that system (3) possesses a distinct ergodic stationary distribution $\tilde{\omega}(\cdot)$, provided that,*

$$\mathcal{R}_0^S = \frac{(\beta_1\phi\lambda + \beta_2)}{\left(\mu + \frac{\varrho_1^2}{2}\right)\left(\phi + \mu + \frac{\varrho_2^2}{2}\right)\left(\pi + \theta + \mu + \frac{\varrho_3^2}{2}\right)} > 1.$$

This expression for $\mathcal{R}_0^S$ aligns with the threshold $\mathcal{R}_0$ of the deterministic system (2) when the effect of white noise is disregarded. This implies that our results build upon and broaden the findings of the deterministic system.

6 Eradication of The Disease

Another major concern in epidemiology is regulating disease dynamics in order to eradicate the disease permanently. Here, we will identify the essential conditions required for disease extinction in the stochastic Model (3). It can be shown that $\chi(t)$ is an integral function over the range $[0,\infty)$. For convenience, we define that,

$$\langle \chi(t) \rangle_t = \frac{1}{t}\int_0^t \chi(s)d\mathscr{B}(s).$$

Lemma 6.1. *Assume $(\mathcal{S}(t),\mathcal{E}(t),I(t),\mathcal{R}(t))$ denote the solution of the system (3) with the beginning values $(\mathcal{S}(0),\mathcal{E}(0),I(0),\mathcal{R}(0)) \in \mathscr{R}_+^4$. Then,*

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\ln \mathcal{S}(t)=0, \quad \limsup_{t\to\infty}\frac{1}{t}\ln \mathcal{E}(t)=0,\\ &\limsup_{t\to\infty}\frac{1}{t}\ln I(t)=0, \quad \limsup_{t\to\infty}\frac{1}{t}\ln \mathcal{R}(t)=0, \quad a.s. \end{aligned} \tag{33}$$

Along with, if $\mu > \dfrac{\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2}{2}$, then,

$$\begin{aligned} &\lim_{t\to\infty}\frac{\int_0^t \mathcal{S}(u)d\mathscr{B}_1(u)}{t}=0, \quad \lim_{t\to\infty}\frac{\int_0^t \mathcal{E}(u)d\mathscr{B}_2(u)}{t}=0,\\ &\lim_{t\to\infty}\frac{\int_0^t I(u)d\mathscr{B}_3(u)}{t}=0, \quad \lim_{t\to\infty}\frac{\int_0^t \mathcal{R}(u)d\mathscr{B}_4(u)}{t}=0, \quad a.s. \end{aligned} \tag{34}$$

Lemma 6.1 will be demonstrated using approaches analogous to those in [16, Lemmas 2.1 and 2.2], so it will not be detailed here.

Specify the parameter as follows,

$$\widehat{\mathcal{R}}_0^S = \frac{(\beta_1\phi+\beta_2)(\phi+\mu)}{\left[\mu(\mu+\phi)^2\left(\pi+\mu+\theta+\dfrac{\varrho_4^2}{2}\right)\right]\wedge\dfrac{\phi^2\varrho_2^2}{2}}.$$

Theorem 6.1. *Assume that $(\mathcal{S}(t),\mathcal{E}(t),I(t),\mathcal{R}(t))$ is a solution of the Model (3) with a starting value of $(\mathcal{S}(0),\mathcal{E}(0),I(0),\mathcal{R}(0)) \in \mathscr{R}_+^4$. If $\widehat{\mathcal{R}}_0^{S^*} < 1$ and also $\mu > \dfrac{\varrho_1^2 \vee \varrho_2^2 \vee \varrho_3^2 \vee \varrho_4^2 \vee}{2}$, then the solution $(\mathcal{S}(t),\mathcal{E}(t),I(t),\mathcal{R}(t))$ of the Model (3) satisfies,*

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}(\ln(\phi\mathcal{E}(t)+(\mu+\phi)I(t))) &\le \frac{\beta_1+\beta_2\phi}{\phi+\mu}-\frac{1}{(\phi+\mu)^2}\\ &\quad\left[(\mu+\phi)^2\left(\pi+\mu+\theta+\frac{\varrho_4^2}{2}\right)\right]\wedge\frac{\phi^2\varrho_2^2}{2}\\ &< 0, \quad a.s. \end{aligned}$$

$\lim_{t\to\infty}\mathcal{R}(t)=0, \quad a.s.$

As $\mathcal{S}(t)$ is distributed, it will eventually converge to the measure with the highest density.

$$f_*(y)=\mathscr{C}y^{-2}y^{-\frac{2\mu}{\varrho_1^2}} \quad e^{-\frac{2\lambda}{\varrho_1^2 y}},$$

where $\mathscr{C}=\left[\dfrac{2\lambda}{\varrho_1^2}^{-\frac{2\mu+\varrho_1^2}{\varrho_1^2}}\Upsilon\left(\dfrac{2\mu+\varrho_1^2}{\varrho_1^2}\right)\right]^{-1}.$

Proof. Consider the SDE,

$$d\widetilde{u} = (\lambda - \mu\widetilde{u})dt + \varrho_1\widetilde{u}d\mathscr{B}_1(t). \tag{35}$$

It is straightforward to verify that (35) has a stationary solution $\widetilde{u}(t)$ with the density given by $f_*(y) = \mathscr{C}y^{-2}y^{-\frac{2\mu}{\varrho_1^2}}e^{-\frac{2\lambda}{\varrho_1^2 y}}$, where $\mathscr{C} = \left[\frac{2\lambda}{\varrho_1^2}^{-\frac{2\mu+\varrho_1^2}{\varrho_1^2}}\Upsilon\left(\frac{2\mu+\varrho_1^2}{\varrho_1^2}\right)\right]^{-1}$.

Following the ergodic theorem,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{S}(s)ds = \int_0^\infty yf_*(y)dy, \quad \text{a.s.} \tag{36}$$

From SDE (36), with help of direct calculation, we get,

$$\int_0^\infty yf_*(y)dy = \mathscr{E}(\widetilde{u}(t)) = \frac{\lambda}{\mu}.$$

Let $\widetilde{u}(t)$ denote the solution to SDE (35) with the starting value $\widetilde{u}(0) = y(0) > 0$. To establish this, we will utilize the comparison theorem for stochastic differential equations from [7]. Therefore, we obtain,

$$\mathcal{S}(t) \le \widetilde{u}(t), \quad \text{a.s.} \tag{37}$$

Additionally, let $\mathscr{P}(t) = \phi\mathcal{E}(t) + (\mu+\phi)I(t)$. According to Ito's formula,

$$\begin{aligned}
d\ln\mathscr{P}(t) &= \left\{\frac{\frac{\phi\beta_1\mathcal{S}(t)I(t)}{1+\alpha I} + \phi\beta_2\mathcal{S}(t)\mathcal{E}(t) - \mu(\phi+\mu)(\pi+\mu+\theta)I(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}\right.\\
&\quad \left. - \frac{\phi^2\varrho_2^2\mathcal{E}^2(t) + (\phi+\mu)^2\varrho_3^2I^2(t)}{2[\phi\mathcal{E}(t)+(\phi+\mu)I(t)]^2}\right\}dt\\
&\quad + \frac{\phi\varrho_2\mathcal{E}(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_2(t) + \frac{(\phi+\mu)\varrho_3I(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_3(t)\\
&\le \frac{\phi(\beta_1+\beta_2)}{\mu(\phi+\mu)}dt - \frac{1}{[\phi\mathcal{E}(t)+(\phi+\mu)\mathcal{E}(t)]^2}\\
&\quad \times\left\{\left[\mu(\phi+\mu)^2(\pi+\mu+\theta) + \frac{1}{2}(\phi+\mu)^2\varrho_3^2\right]I^2(t) + \frac{1}{2}\phi^2\varrho_2^2\mathcal{E}^2(t)\right\}dt\\
&\quad + \frac{\phi\varrho_2\mathcal{E}(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_2(t) + \frac{(\phi+\mu)\varrho_3I(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_3(t)\\
&\le \frac{\phi(\beta_1+\beta_2)}{\mu(\phi+\mu)}dt - \frac{1}{2\mu(\phi+\mu)^2}\left\{\left[\mu(\phi+\mu)^2\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right)\right]\wedge\frac{\phi^2\varrho_2^2}{2}\right\}dt\\
&\quad + \frac{\phi\varrho_2\mathcal{E}(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_2(t) + \frac{(\phi+\mu)\varrho_3I(t)}{\phi\mathcal{E}(t)+(\phi+\mu)I(t)}d\mathscr{B}_3(t).
\end{aligned} \tag{38}$$

After integrating both sides of (38) over the interval from 0 to t and then dividing by t, we derive,

$$\begin{aligned}
\frac{\ln\mathscr{P}(t)}{t} - \frac{\ln\mathscr{P}(0)}{t} &\le \frac{\phi(\beta_1+\beta_2)}{\mu(\phi+\mu)} - \frac{1}{2\mu(\phi+\mu)^2}\left\{\left[\mu(\phi+\mu)^2\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right)\right]\wedge\frac{\phi^2\varrho_2^2}{2}\right\}\\
&\quad + \frac{\phi\varrho_2}{t}\int_0^t\frac{\mathcal{E}(\mathcal{S})}{\phi\mathcal{E}(\mathcal{S})+(\phi+\mu)I(\mathcal{S})}d\mathscr{B}_2(\mathcal{S})\\
&\quad + \frac{\mu(\phi+\mu)\varrho_3}{t}\int_0^t\frac{I(\mathcal{S})}{\phi\mathcal{E}(\mathcal{S})+(\phi+\mu)I(\mathcal{S})}d\mathscr{B}_3(\mathcal{S}).
\end{aligned} \tag{39}$$

Taking both sides of the limit supremum (39) to combine Lemma 6.1 and $\widehat{\mathcal{R}_0}^S < 1$, we can obtain,

$$\limsup_{t\to\infty} \frac{\ln \mathscr{P}(t)}{t} \leq \frac{\phi(\beta_1+\beta_2)}{\mu(\phi+\mu)} - \frac{1}{2\mu(\phi+\mu)^2}\left\{\left[\mu(\phi+\mu)^2\left(\pi+\mu+\theta+\frac{\varrho_3^2}{2}\right)\right]\wedge\frac{\phi^2\varrho_2^2}{2}\right\}$$
$$< 0, \quad \text{a.s.}$$

Consequently,

$$\lim_{t\to\infty} \mathcal{E}(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} I(t) = 0, \quad \text{a.s.}$$

Furthermore, in case of any sufficiently small $\phi > 0$, there exist t_0 and a set $\Omega_\phi \subset \Omega$ such that $\mathcal{P}(\Omega_\phi) > 1-\phi$ and $\beta\mathcal{S}I \leq \phi\mathcal{S}$ for $t \geq t_0$ and $\omega \in \Omega_\phi$. Thus, the expression becomes,

$$\left(\lambda - \frac{(\beta_1+\beta_2)\mathcal{S}}{1+\alpha I} - \mu\mathcal{S} + \delta\mathcal{R}\right)dt + \varrho_1\mathcal{S}d\mathscr{B}_1(t) \leq d\mathcal{S}$$
$$\leq (\Lambda - \mu\mathcal{S})dt + \varrho_1\mathcal{S}d\mathscr{B}_1(t). \tag{40}$$

Equation (36) can be integrated from 0 to t and combined with Lemma 6.1 to yield,

$$\liminf_{t\to\infty} \langle\mathcal{S}\rangle_t \geq \frac{\lambda}{\mu+\gamma}, \quad \text{a.s.} \tag{41}$$

There is an arbitrariness to γ,

$$\liminf_{t\to\infty} \langle\mathcal{S}\rangle_t \geq \frac{\lambda}{\mu}, \quad \text{a.s.} \tag{42}$$

Thus from (36), (41) and (42), we get,

$$\lim_{t\to\infty} \langle\mathcal{S}\rangle_t \geq \frac{\lambda}{\mu}, \quad \text{a.s.} \tag{43}$$

According to (36) and (37), the process $\mathcal{S}(t)$ distribution converges to the measure that is represented by the density f_*.

Using the Model (3), it is straightforward to observe that if $\lim_{t\to\infty} I(t) = 0$ a.s. then, $\lim_{t\to\infty} \mathcal{R}(t) = 0$ a.s. as well. The proof is now finished. □

Remark 6.1. *Theorem 6.1 demonstrates that if,*

$$\widehat{\mathcal{R}_0}^S = \frac{(\beta_1\phi+\beta_2)(\phi+\mu)}{\left[\mu(\mu+\phi)^2\left(\pi+\mu+\theta+\frac{\varrho_4^2}{2}\right)\right]\wedge\frac{\phi^2\varrho_2^2}{2}} < 1.$$

The disease will become extinct. It is important to note that when $\widehat{R}_0^S < 1$, a higher intensity of white noise facilitates the extinction of the disease. Therefore, controlling the outbreak can be achieved by adjusting the level of environmental noise.

7 Discussion and Numerical Simulations

In this work, we investigated the processes of illness extinction and persistence inside the framework of Model (3). We shall run multiple numerical simulations to verify the validity of

our conclusions. The Milstein method [7] is used numerically to underline the variations in $\mathcal{R}_0$ behavior between the deterministic and stochastic forms of the SEIRS epidemic model. The SDE Model (3) is formulated as follows:

$$\begin{aligned}
\mathcal{S}_{i+1} &= \mathcal{S}_i \left(\lambda - \mathcal{S}_i \frac{\beta_1 I_i}{1+\alpha_1 I_i} - \mathcal{S}_i \beta_2 \mathcal{E}_i - \mu \mathcal{S}_i + \delta \mathcal{R}_i \right) \Delta t + \varrho_1 \mathcal{S}_i \sqrt{\Delta t} \gamma_i + \frac{\varrho_1^2}{2} \mathcal{S}_i (\gamma_i^2 - 1) \Delta t, \\
\mathcal{E}_{i+1} &= \mathcal{E}_i \left(\mathcal{E}_i \frac{\beta_1 I_i}{1+\alpha_1 I_i} + \mathcal{S}_i \beta_2 \mathcal{E}_i - (\mu + \phi) \mathcal{E}_i \right) \Delta t + \varrho_2 \mathcal{E}_i \sqrt{\Delta t} \eta_i + \frac{\varrho_2^2}{2} \mathcal{E}_i (\eta_i^2 - 1) \Delta t, \\
I_{i+1} &= I_i \left(\phi \mathcal{E}_i - (\pi + \mu + \theta) I_i \right) \Delta t + \varrho_3 I_i \sqrt{\Delta t} \zeta_i + \frac{\varrho_3^2}{2} I_i (\zeta_i^2 - 1) \Delta t, \\
\mathcal{R}_{i+1} &= \mathcal{R}_i \left(\pi I_i - (\mu + \delta) \mathcal{R}_i \right) \Delta t + \varrho_4 \mathcal{R}_i \sqrt{\Delta t} \psi_i + \frac{\varrho_4^2}{2} \mathcal{R}_i (\psi_i^2 - 1) \Delta t.
\end{aligned}$$

In this scenario, $\gamma_i, \eta_i, \zeta_i, \psi_i$ for $(i = 1, \ldots, 4,)$ are random variables that follow independent Gaussian distributions, denoted as $\mathcal{N}$-distributions.

Example 7.1. *Assume that Model (3) has the following parameters:* $\lambda = 5, \beta_1 = 0.9, \beta_2 = 0.05, \phi = 0.5, \mu = 0.3, \alpha = 2, \delta = 0.1, \theta = 0.1, \pi = 0.5, (\varrho_1, \varrho_2, \varrho_3, \varrho_4) = (0.1, 0.1, 0.1, 0.1)$ *as well as the starting condition is* $(\mathcal{S}(0), \mathcal{E}(0), I(0), \mathcal{R}(0)) = (0.7, 0.2, 0.1, 0)$. *We can obtain that,*

$$\mathcal{R}_0^S = \frac{(\beta_1 \phi \lambda + \beta_2)}{\left(\mu + \frac{\varrho_1^2}{2}\right)\left(\phi + \mu + \frac{\varrho_2^2}{2}\right)\left(\pi + \theta + \mu + \frac{\varrho_3^2}{2}\right)} = 10.3510 > 1.$$

The graphical representations of Example 7.1 can be found in Figures 1, 3, and 5, offering a detailed visualization of the model's behavior and facilitating a deeper understanding of the system's dynamics through different perspectives.

Example 7.2. *According to the Model (3), the following parameters are considered:* $\lambda = 2, \beta_1 = 0.37, \beta_2 = 0.05, \phi = 0.5, \mu = 0.3, \alpha = 2, \delta = 0.1, \theta = 0.1, \pi = 0.5, (\varrho_1, \varrho_2, \varrho_3, \varrho_4) = (0.1, 0.1, 0.1, 0.1)$ *as well as the starting condition* $(\mathcal{S}(0), \mathcal{E}(0), I(0), \mathcal{R}(0)) = (0.7, 0.2, 0.1, 0)$. *It is easy to find the basic reproduction number,*

$$\begin{aligned}
\mathcal{R}_0 &= \frac{\beta_1 \phi \mathcal{S}_0}{(\mu + \phi)(\pi + \mu + \theta)} + \frac{\beta_2}{\mu + \phi} = 7.2740 > 1, \quad \textit{and} \\
\widehat{\mathcal{R}}_0^S &= \frac{(\beta_1 \phi + \beta_2)(\phi + \mu)}{\left[\mu(\mu + \phi)^2 \left(\pi + \mu + \theta + \frac{\varrho_4^2}{2}\right)\right] \wedge \frac{\phi^2 \varrho_2^2}{2}} = 0.9793 < 1.
\end{aligned}$$

The graphical representations in Figures 2, 4, and 6 offer a comprehensive illustration of Example 7.2, highlighting the trends and interactions within the system, and providing valuable insights into the model's behavior.

Consequently, the disease may disappear due to white noise, as it may lead to its extinction. Meanwhile, the disease has existed for deterministic models (2) since $\mathcal{R}_0 > 1$. In this way, numerical simulations validate the theoretical results.

The Figures 1–6 illustrate the fluctuation of susceptible, exposed, and infected groups over time in relation to the stationary distribution and extinction of the diseases. These observations are made in both deterministic and stochastic systems. Additionally, curve comparisons of the same values indicate a closer approximation to the reality of both deterministic and stochastic numerical simulation of the Model (3).

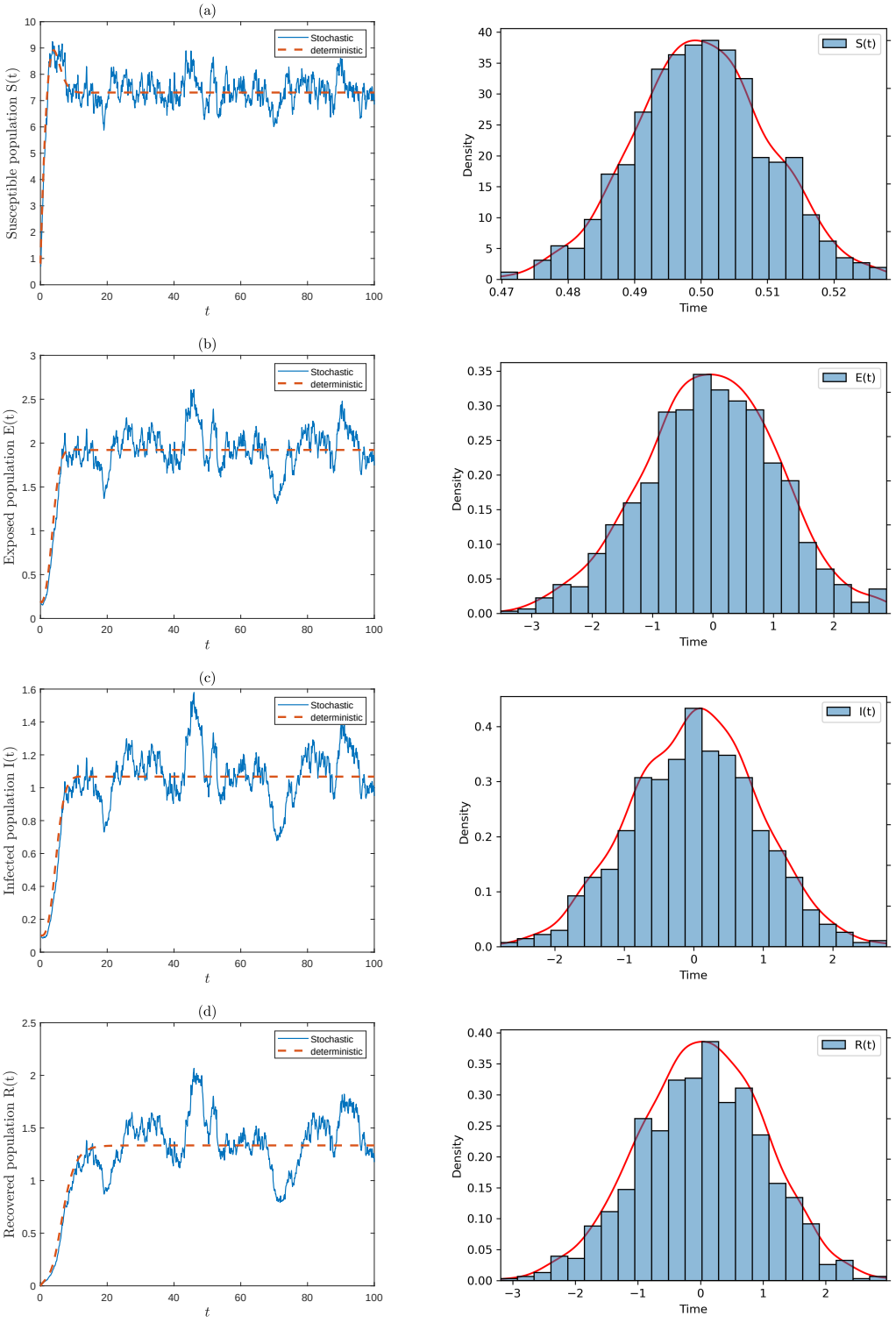

Figure 1: The comparison of solutions for $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$ between the deterministic and stochastic systems under the condition $R_0^S > 1$ is illustrated. The histograms on the right column display the probability density functions for $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$.

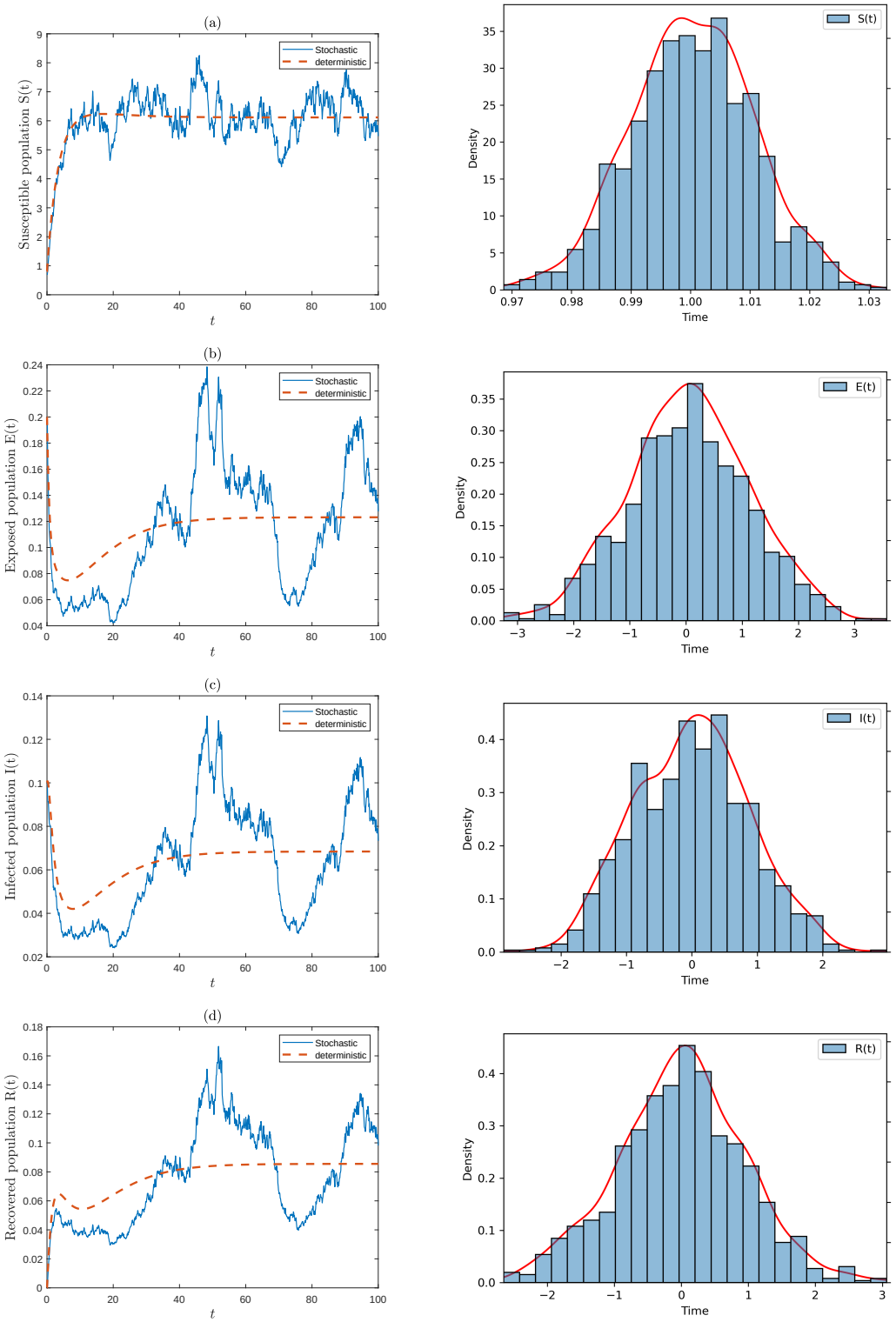

Figure 2: The comparison of solutions for $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$ between the deterministic and stochastic systems under the condition $\widehat{R}_0^S < 1$ is illustrated. The histograms on the right column display the probability density functions for $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$.

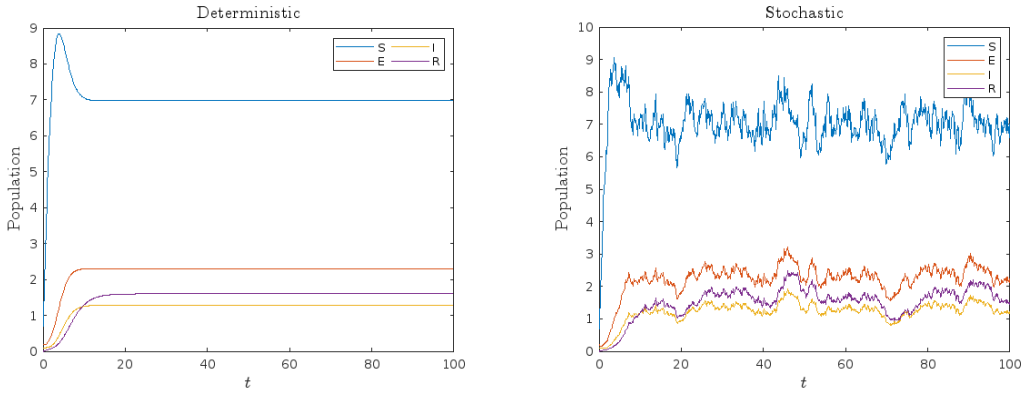

Figure 3: The behavior of each compartment in both the Deterministic and Stochastic systems is analyzed. When $\mathcal{R}_0^S > 1$.

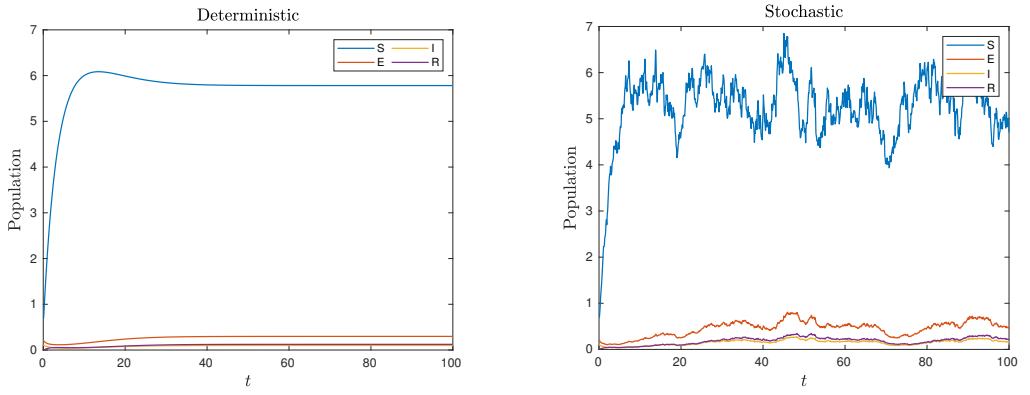

Figure 4: The behavior of each compartment in both the Deterministic and Stochastic systems is analyzed. When $\widehat{\mathcal{R}}_0^S < 1$.

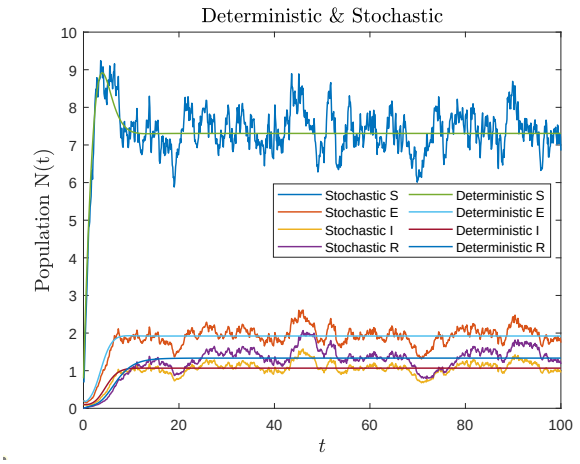

Figure 5: Comparison of solutions on $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$: for each class in Deterministic & Stochastic system with $\mathcal{R}_0^S > 1$.

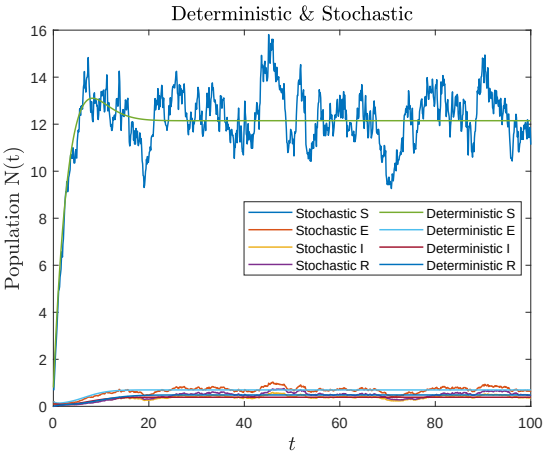

Figure 6: Comparison of solutions on $\mathcal{S}(t)$, $\mathcal{E}(t)$, $I(t)$, and $\mathcal{R}(t)$: for each class of the Deterministic & Stochastic systems when $\widehat{\mathcal{R}}_0^S < 1$.

8 Conclusion

This paper has investigated the comparison between the long-term behavior of a stochastic SEIRS epidemic model and the saturation incidence rate $g(I)\mathcal{S} = \dfrac{\beta_1 \mathcal{S} I}{1+\alpha I}$, focusing on the extended characteristics and dynamics of the stochastic SEIRS model. Relevant research is specifically cited in [21]. When a virus enters the human body, it does not immediately cause an infection; instead, it may remain latent for a certain period. Since the incubation period varies across infectious diseases, it is crucial to study models that incorporate this phase. Based on the standard and saturated incidence rates, when the random perturbation coefficient is sufficiently small and the parameters $\beta_1 \mathcal{S} I$ and $\beta_2 \mathcal{S}\mathcal{E}$ are appropriately regulated, the system's overall dynamics become more predictable. This improved understanding supports the adoption of the most effective strategies to control and mitigate the spread of infectious diseases. The stochastic Model (3) necessitates $\mathcal{R}_0^S > 1$ in order to establish the presence of a stationary distribution, while $\widehat{\mathcal{R}}_0^S < 1$ signifies the eradication of the disease. The mandatory conditions for eradicating the disease have also been formulated. The intensity of environmental noise also catalyses the persistence of infectious diseases. At the end, our findings are validated using numerical simulations.

There are several intriguing and transparent topics that warrant further exploration. Factors such as sudden climate change, weather warming, cooling, or wetting and drying can significantly influence disease proliferation. It is important to highlight, however, that the methods employed in this study can be extended to other epidemic models of interest. Notable examples include SEIQR, SEIQVR, and other variations. This adaptability paves the way for continued research in the future, offering a robust framework for analyzing and managing a wide range of infectious diseases under varying environmental conditions.

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